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The Square Root of 2 to 1,000,000 Decimals

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Abstract. The square root of 2 has been calculated to 1,000,000 decimals on a large-scale digital computer and the result has been verified. The calculation was based on a specially developed algorithm for square roots which does not appear to have been used in previous computations of this type.

1. Introduction. A summary of extended-length computations of the square root of 2 is given in the following table.

Author	*Machine*	*Date*	*Precision*
R. Coustal [1]		1950	1032 D
H. S. Uhler [2]	Desk	1950	1544 D
K. Takahashi and M. Sibuya [3]	HIPAC-103	1966	14000 D
M. Lal [4]	IBM-1620	1967	19600 D
M. Lal [5]	IBM-1620	1967	39000 D
M. Lal and W. F. Lunnon [6]	ATLAS	1967	100,000 D

The computations of R. Coustal and H. S. Uhler made use of binomial series expansions. Takahashi and Sibuya employed an iterative method based on the formula $x_{k+1} = x_k(1.5 - 0.5nx_k^2)$ which requires only multiplications and additions. In his first calculation, M. Lal employed a special method which yields one digit at a time. In the later calculations the Newton method was employed to extend the original result.

This method, which is by far the most widely used algorithm for obtaining square roots in modern electronic computers, depends on iterations of the formula

$$(1) \qquad a_{n+1} = \frac{a_n + N/a_n}{2}, \qquad n = 0, 1, 2, \cdots,$$

where a_0 is an initial approximation to $\sqrt{N}$. If the initial approximation is suitably chosen, the process converges quickly and accurate single- or even double-precision approximations to $\sqrt{N}$ are obtained after only a few iterations. However, if more extended multiple-precision approximations to $\sqrt{N}$ are sought, the computation time increases rapidly because of the times required for dividing N by a many-digit number. Generally, the time required for floating-point division on modern electronic computers compared to floating-point multiplication is at least twice as much for double-

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precision computations. The comparison is usually even less favorable for division in extended multiple-precision computation. Overall, the use of (1) to obtain many-decimal approximations appears efficient only if good initial approximations are already available and a small number of iterations are required to obtain the final result.

2. A Quadratically Converging Algorithm. The algorithm actually employed for the calculation of the square root of 2 to 1,000,000 decimals depends on the generation of solutions of the Pell (Fermat) equation $P^2 - NQ^2 = 4$ by means of recurrence relations involving multiplications, and the approximation of $\sqrt{N}$ by a suitable ratio P/Q. As is well known, if N is a nonsquare positive integer, this equation has an infinite number of positive integer solutions which can be obtained from the convergents of the continued fraction expansion of $\sqrt{N}$. (See, e.g., Nagell [7, pp. 204 ff.].) In particular, suitable starting values (P_0, Q_0) for the recurrence relations in the following theorem can be obtained from the continued fraction expansion.

THEOREM 1. *Let (P_0, Q_0) denote a solution in positive integers of the Pell equation $P^2 - NQ^2 = 4$ where N is a nonsquare positive integer, and let*

$$(2) \qquad P_{n+1} = P_n^2 - 2, \qquad Q_{n+1} = P_n Q_n, \qquad n = 0, 1, 2, \cdots .$$

Then (P_n, Q_n) is a solution of the Pell equation, and as $n \to \infty$, $P_n/Q_n \to \sqrt{N}$. The sequence $\{P_n/Q_n\}$ is equivalent to the sequence $\{a_n\}$ obtained from (1) *with the initial approximation $a_0 = P_0/Q_0$.*

Proof. From (2),

$$P_{n+1}^2 - NQ_{n+1}^2 = (P_n^2 - 2)^2 - NP_n^2Q_n^2 = P_n^2(P_n^2 - NQ_n^2 - 4) + 4$$

and it follows by mathematical induction that (P_n, Q_n) is a solution of the Pell equation. It also follows from this and (2) that

$$\frac{P_n \pm Q_n\sqrt{N}}{2} = \left(\frac{P_0 \pm Q_0\sqrt{N}}{2}\right)^{(2^n)}.$$

Solving for P_n and Q_n and dividing, one finds

$$(3) \qquad \frac{P_n}{Q_n} = \sqrt{N}\left[\frac{1 + \alpha^{(2^n)}}{1 - \alpha^{(2^n)}}\right] \quad \text{where } \alpha = \frac{P_0 - Q_0\sqrt{N}}{P_0 + Q_0\sqrt{N}}.$$

Since $|\alpha| < 1$, it follows that as $n \to \infty$, $P_n/Q_n \to \sqrt{N}$ and the convergence is quadratic. Moreover, (1) is satisfied with $a_0 = P_0/Q_0$, for

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{2P_n^2 - 4}{2P_nQ_n} = \frac{P_n^2 + NQ_n^2}{2P_nQ_n} = \frac{(P_n/Q_n) + N(Q_n/P_n)}{2}.$$

Bounds for the difference $(P_n/Q_n - \sqrt{N})$ can readily be found . For since $(P_n - Q_n\sqrt{N})(P_n + Q_n\sqrt{N}) = 4$, it follows that

$$\frac{P_n}{Q_n} - \sqrt{N} = \frac{4}{Q_n(P_n + Q_n\sqrt{N})}.$$

Evidently, $P_n > Q_n\sqrt{N}$ and $NQ_n^2/P_n < Q_n$. Hence

$$\text{(4)} \qquad \frac{2}{P_nQ_n} < \frac{P_n}{Q_n} - \sqrt{N} < \frac{4P_n}{Q_n(P_n^2 + NQ_n^2)}.$$

But $P_nQ_n = Q_{n+1}$ and $P_n^2 + NQ_n^2 = 2P_{n+1}$. Thus

$$\text{(4')} \qquad \frac{2}{Q_{n+1}} < \frac{P_n}{Q_n} - \sqrt{N} < \frac{2P_n}{P_{n+1}Q_n}$$

and the difference between the upper and lower bounds is $4/Q_{n+2}$.

3. Computational Considerations. Given an initial (P_0, Q_0) which is a solution of $P^2 - NQ^2 = 4$, how large must n be in (2) so that P_n/Q_n approximates $\sqrt{N}$ with a specified accuracy?

In (3) let $1/\beta$ denote the required accuracy. Then

$$\frac{P_n}{Q_n} - \sqrt{N} = \sqrt{N}\left[\frac{1 + \alpha^{(2^n)}}{1 - \alpha^{(2^n)}} - 1\right] = \left[\frac{2\alpha^{(2^n)}\sqrt{N}}{1 - \alpha^{(2^n)}}\right] \leqq \frac{1}{\beta}.$$

Solving for n from the inequality on the right, one finds that

$$\text{(5)} \qquad n \geqq \frac{\log\log(2\beta\sqrt{N} + 1) - \log\log((P_0 + Q_0\sqrt{N})/2)}{\log 2} - 1.$$

For the computation of $\sqrt{2}$, the values

$$P_0 = 6726, \qquad Q_0 = 4756, \qquad \beta = 10^{1,000,000}$$

were chosen. From (5) it was found that $n = 17$. The integers P_{17} and Q_{17} each contain 501,712 decimal digits.

Although, as has been pointed out above, the algorithm of Theorem 1 is essentially equivalent to the Newton method for obtaining square roots, it appears from a computational standpoint to have certain advantages for many-decimal approximations. These may be summarized as follows:

(i) Divisions in the Newton method are, except for one division as the last step in the application of the algorithm, replaced by multiplications. Specifically, at each stage a division in the Newton method is replaced by about one and a half multiplications in (2). For, as is well known, to calculate the $m \times m$ symmetric square array of partial products obtained by multiplying the m(computer)-word number P_n by itself, it is only necessary to compute $m(m + 1)/2$ partial products—corresponding to the terms in the square array which are on or above the main diagonal instead of m^2 partial products. If m is large, this is about $m^2/2$ multiplications.

(ii) P_n/Q_n is equivalent to a convergent in the continued fraction expansion of $\sqrt{N}$ and thus has the well-known optimum property of rational approximations to $\sqrt{N}$ of such convergents.

(iii) Integer arithmetic is used at each stage, so that computational operations are made more convenient, e.g., there is no loss of significance as occurs in the truncation of decimal fractions. Moreover, the fact that (P_n, Q_n) is a solution of a Pell equation can be exploited to provide checks for the accuracy of the computations at each step.

Higher-order algorithms which converge more rapidly than that of Theorem 1 have been considered. But the gain in rapidity of convergence is obtained at the cost of increasing complexity of the recurrence formulas analogous to (2), and such algorithms do not appear to be advantageous from a practical standpoint.

4. Results. The calculation of the square root of 2 by the algorithm of Theorem 1 was carried out on Columbia University's IBM 360/91 computer at odd times spread out over more than a year. The computer program which was written made extensive use of a multiple-precision floating-point arithmetic subroutine developed by J. R. Ehrman [8]. The calculation, which took about 47.5 hours, consisted of the following steps:

The generation of (P_{17}, Q_{17}) from (2) with the starting values $P_0 = 6726, Q_0 = 4756$, the division $P_{17} \div Q_{17}$ carried out to the equivalent of more than a million decimal places, the conversion of this quotient from hexadecimal to decimal form, and finally the verification of this approximation to $\sqrt{2}$ by squaring it and comparing it with $2 = 1.999 \cdots$.

The lengthiest operations were the conversion from the hexadecimal to the decimal form and the division. The number obtained by squaring the approximation in the verification showed one, decimal point followed by 1,000,082 nines, so that the accuracy of the approximation is guaranteed to this number of places.

The computer printout, which has been deposited in the UMT file, is in the form of 200 pages, each containing 5000 decimal digits, and a final page on which the first 82 digits are correct.

An analysis of the pseudo-random characteristics of the approximation will be made and the results published.

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